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Efficient Augmentation to Construct $(\sigma+1)$-Edge-Connected Simple Graphs

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Abstract: The unweighted *k-edge-connectivity augmentation problem* (kECA for short) is defined by "Given a σ-edge-connected graph $G=(V,E)$, find an edge set E' of minimum cardinality such that $G'=(V,E\cup E')$ is $(\sigma+\delta)$-edge-connected and $\sigma+\delta=k$", where E' is called a *solution* to the problem. Let kECA(S,SA) denote kECA such that both G and G' are simple.

The subject of the present paper is $(\sigma+1)$ECA(S,SA) (or kECA(S,SA) with $k=\sigma+1$). Let $\mathcal{M}$ be any maximum matching of a certain graph $R(G)$ whose vertex set V_R consists of vertices representing all leaves of G. From $\mathcal{M}$ we obtain an edge set E'_0, with $|E'_0|=|\mathcal{M}|$, such that each edge connects vertices in distinct leaves of G. Let $\mathcal{L}_1$ be the set of leaves to be created by adding E'_0 to G, and $\mathcal{K}_1$ the set of remaining leaves of G.

The main result is to propose two $O(\sigma^2|V|\log(|V|/\sigma)+|E|+|V_R|^2)$ time algorithms for finding the following solutions: (1) an optimum solution if G has at least $2\sigma+6$ leaves or if $|\mathcal{L}_1|\le|\mathcal{K}_1|$ and G has less than $2\sigma+6$ leaves; (2) a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1|>|\mathcal{K}_1|$ and G has less than $2\sigma+6$ leaves.

Keywords: Edge-connectivity, minimum cuts, polynomial time algorithms, augmentation problem, maximum matchings.

1 Introduction

The unweighted *k-edge-connectivity augmentation problem* (kECA for short) is described as follows: "Given a σ-edge-connected graph $G=(V,E)$, find an edge set E' of minimum cardinality such that $G'=(V,E\cup E')$ is $(\sigma+\delta)$-edge-connected and $\sigma+\delta=k$." We often denote G' as $G+E'$, and E' is called a *solution* to the problem. Let kECA(*,**) denote kECA with the following restriction (i) and (ii) on G and E', respectively: (i) * is set to S if G is required to be simple, and * is left to mean that G may be a multiple graph; (ii) ** is set to MA if creation of new multiple edges in constructing G' is allowed, and is set to SA otherwise. In kECA(*,SA), if G is simple then so is G', or if G has multiple edges then any multiple edge of G' exists in G. As for kECA, kECA(*,MA) has mainly been discussed so far. See [3, 5, 7, 8, 12, 13, 21–24] for the results. It is natural for us to assume that $|V|\ge\sigma+2$ in $(\sigma+1)$ECA(S,SA): in $(\sigma+1)$ECA(*,SA), we may have $|V|\le\sigma+1$.

As related results, kECA(S,SA) for G having no edges was first discussed in [6], where the problem that is more general than kECA(S,SA) is considered. An $O(|V|+|E|)$ algorithm for 2ECA(S,SA) can be obtained by slightly modifying the one given in [3] for 2ECA(*,MA). As for 3ECA(*,SA), [24] proposed an $O(|V|+|E|)$ algorithm for 3ECA(*,MA), and showed that if $|V|\ge 4$ then this algorithm finds an optimum solution to 3ECA(*,SA). Concerning $(\sigma+1)$ECA(S,SA) with $|V|\ge\sigma+2$ for $\sigma\in\{3,4\}$, [15] proposed an $O(|V|\log|V|+|E|)$ algorithm. Other related results have been reported in [14, 16]. T. Jordán showed in [10] that kECA(S,SA) is NP-hard in general, and [2] proposed an $O(|V|^4)$ algorithm for kECA(S,SA) for any fixed k.

The subject of the present paper is $(\sigma+1)$ECA(S,SA), that is, kECA(S,SA) with $k=\sigma+1$. Let $\mathcal{M}$ be any maximum matching of the *leaf-graph* $R(G)$ whose vertex set V_R consists of vertices representing all leaves of G. (The definition of $R(G)$ is going to be given later). From $\mathcal{M}$ we obtain a certain edge set E'_0, with $|E'_0| = |\mathcal{M}|$, such that each edge connects vertices in distinct leaves of G. Let $\mathcal{L}_1$ be the set of leaves to be created by adding E'_0 to G, and $\mathcal{K}_1$ the set of remaining leaves of G.

The main result of the paper is to propose two $O(\sigma^2|V|\log(|V|/\sigma) + |E| + |V_R|^2)$ time algorithms for finding the following solutions for $(\sigma+1)$ECA(S,SA):

(1) an optimum solution if G has at least $2\sigma+6$ leaves or if $|\mathcal{L}_1| \le |\mathcal{K}_1|$ and G has less than $2\sigma+6$ leaves;
(2) a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1| > |\mathcal{K}_1|$ and G has less than $2\sigma+6$ leaves.

A central concept in solving kECA is a *t-edge-connected component* of G: a maximal set of vertices such that G has at least t edge-disjoint paths between any pair of vertices in the set [23]. A t-edge-connected component whose degree (the number of edges connecting vertices in the set to those outside of it) is equal to the edge-connectivity of G is called a *leaf*. Although $(\sigma+1)$ECA(S,SA) can be solved almost similarly to general kECA(*,MA), the only difference is that the augmenting step has to choose a pair of leaves, each containing a vertex such that they are not adjacent in G. (Such a pair of leaves is called a *nonadjacent pair*.) This requires addition of some other characteristics or processes in finding solutions by means of structural graphs: a structural graph is introduced in [11], and is used as a useful tool that reduces time complexity in finding a solution to kECA(*,MA) in [7, 13].

This paper adopts the operation, called *edge-interchange*, in finding a solution, where it was introduced in [21, 22] in order to reduce time complexity of [23]. A set of two nonadjacent pairs of leaves is called a *D-combination* if they are disjoint. The augmenting step in solving $(\sigma+1)$ECA(S,SA) repeats both choosing a nonadjacent pair of leaves and enlarging a $(\sigma+1)$-edge-connected component by means of edge-interchange (or an analogous operation). Hence obtaining an optimum solution requires finding a maximum set of nonadjacent pairs of leaves such that any two members in the set form a D-combination and, therefore, this is reduced to finding a maximum matching of the leaf-graph $R(G)$ of G. The point of $(\sigma+1)$ECA(S,SA) is that a solution E' is closely related to a maximum matching $\mathcal{M}$ of $R(G)$.

The paper is organized as follows. Basic definitions and several basic results on σ-edge-connected componets and leaf-graphs are given in Section 2. In Section 3, results on maximum matchings of leaf-graphs are briefly mentioned. Edge-interchange operation is explained in Section 4. Section 5 discusses $(\sigma+1)$ECA(S,SA) when G has less than $2\sigma+6$ leaves, and Section 6 considers $(\sigma+1)$ECA(S,SA) when G has at least $2\sigma+6$ leaves.

All proofs are omitted becase of space limitation. The early version appeared in [19].

2 Preliminaries

2.1 Basic definitions

Technical terms not specified here can be identified in [1, 4, 9, 20]. An *undirected graph* $G = (V(G), E(G))$ consists of a finite and nonempty set of vertices $V(G)$ and a finite set of undirected edges $E(G)$, where $V(G)$ and $E(G)$ are often denoted as V and E, respectively. An edge e incident upon two vertices u, v in G is denoted by $e = (u, v)$ unless any confusion arises. We denote $V(e) = \{u, v\}$, or generally $V(K) = \{u, v \in V | (u, v) \in K\}$ for a subset $K \subseteq E$. For disjoint sets $X, X' \subset V$, we denote $(X, X'; G) = \{(u, v) \in E | u \in X \text{ and } v \in X'\}$, where it is often written as (X, X') if G is clear from the context. We denote $d_G(X) = |(X, \overline{X}; G)|$. This is called the *degree* of X (in G). We set $d_G(S) = 0$ if $S = \emptyset$. If $X = \{v\}$ then $d_G(\{v\})$ is denoted simply as $d_G(v)$ and is the total number of edges (v, v'), $v' \neq v$, incident upon v. We often denote $d_G(S)$ as $d(S)$ if G is clear from the context. A path between vertices u and v is often called a *(u,v)-path* and denoted by $P_G(u, v)$, and is often written as $P(u, v)$ if G is clear from the context. For two vertices u, v of G, let $\lambda(u, v; G)$, or simply $\lambda(u, v)$, denote the maximum number of pairwise edge-disjoint paths between u and v.

For a set $X \subseteq V$, let $G[X]$ denote the subgraph having X as its vertex set and $\{(u, v) \in E | u, v \in X\}$ as its edge set. $G[X]$ is called the *subgraph* of G *induced* by X (or the *induced subgraph* of G by X). *Deletion* of $X \subseteq V$ from G is to construct $G[V - X]$, which is often denoted as $G - X$. If

$X = \{v\}$ then we often denote $G-v$ for simplicity. Deletion of $Q \subseteq E$ from G defines a spanning subgraph of G, denoted by $G - Q$, having $E - Q$ as its edge set. If $Q = \{e\}$ then we denote $G - e$. For a set E' of edges such that $E' \cap E = \emptyset$, let $G + E'$ denote the graph $(V, E \cup E')$. If $E' = \{e\}$ then we denote $G + e$.

Let $K \subseteq E$ be any minimal set such that $G - K$ has more components than G. K is called a *separator* of G, or in particular a (X, Y)-separator if any vertex of X and any one of Y are disconnected in $G - K$. If $X = \{u\}$ or $Y = \{v\}$ then it is denoted as a (u, Y)-separator or a (X, v)-separator, respectively. A *minimum* (X, Y)*-separator* K of G is a (X, Y)-separator of minimum cardinality. Such K is often called an (X, Y)-cut or an $|K|$-cut. It is known that a (u, v)-cut K has $|K| = \lambda(u, v; G)$. A *minimum separator* K of G is a separator of minimum cardinality among all separators of G, and $|K|$ is called the *edge-connectivity* (denoted by σ) of G; particularly we call such $K \subseteq E$ a *minimum cut* (of G). G is said to be *k-edge-connected* if $\lambda(G) \geq k$. A *k-edge-connected component* (k-component, for short) of G is a subset $S \subseteq V$ satisfying the following (a) and (b): (a) $\lambda(u, v; G) \geq k$ for any pair $u, v \in S$; (b) S is a maximal set that satisfies (a). Let $\Gamma_G(k)$ denote the set of all k-components of G. In a graph G with $\lambda(G) = \sigma$, a $(\sigma + 1)$-component S with $d_G(S) = \sigma$ is called a *leaf* $(\sigma+1)$*-component* of G (or a leaf of G, for short). It is known that $\lambda(G) \geq k$ if and only if V is a k-component. Note that distinct k-components are disjoint sets. Each 1-component is often called a *component.*

Note that we assume that $|V| \geq \sigma + 2$ in $(\sigma + 1)$ECA(S,SA), the subject of the paper.

A *cactus* is an undirected connected graph in which any pair of cycles share at most one vertex. A *structural graph* $F(G)$ of G with $\lambda(G) = \sigma$ is a representation of all minimum cuts of G and is introduced in [11]. We use the term "nodes of $F(G)$" to distinguish them from vertices of G. $F(G)$ is an edge-weighted cactus of $O(|V|)$ nodes and edges such that each tree edge (an edge which is a bridge in $F(G)$) has weight $\lambda(G)$ and each cycle edge (an edge included in any cycle) has weight $\lambda(G)/2$. Let $F(G)$ be a structural graph of G. Particularly if σ is odd then $F(G)$ is a weighted tree. (Examples of G and $F(G)$ will be given in Figs. 1 and 2.) Each vertex in G maps to exactly one node in $F(G)$, and $F(G)$ may have some other nodes, call *empty nodes*, to which no vertices of G are mapped. Let $\epsilon(G) \subseteq V(F(G))$ denote the set of all empty nodes of $F(G)$. Note that any minimum cut of G is represented as either a tree edge or a pair of two cycle edges in the same cycle of $F(G)$, and vice versa. Let ρ: $V \rightarrow V(F(G)) - \epsilon(G)$ denote this mapping. We use the following notations: $\rho(X) = \{\rho(v) | v \in X\}$ for $X \subseteq V$, and $\rho^{-1}(Y) = \{v \in V | \rho(v) \in Y\}$ for $Y \subseteq V(F(G))$. $\rho(\{v\})$ or $\rho^{-1}(\{v\})$ is written as $\rho(v)$ or $\rho^{-1}(v)$, respectively, for notational simplicity. For any cut $(X, V(F(G)) - X; F(G))$, if summation of weights of all edges contained in the cut is equal to σ then $(\rho^{-1}(X), V - \rho^{-1}(X); G)$ is a σ-cut of G. Note that the cut of $F(G)$ consists of either one tree edge or a pair of two cycle edges in the same cycle of $F(G)$. Conversely, for any σ-cut $(X, V - X; G)$, $F(G)$ has at least one cut $(Y, V(F(G)) - Y; G)$ in which summation of weight of all edges contained in the cut is equal to σ, where Y is a node set of $F(G)$ such that $\rho(X) = Y - \epsilon(G)$. Each $(\sigma + 1)$-component S of G is represented as a vertex $\rho(S) \in V(F(G)) - \epsilon(G)$ in $F(G)$, and, for any vertex $v \in V(F(G)) - \epsilon(G)$, $\rho^{-1}(v)$ is a $(\sigma + 1)$-component of G. For $v \in V(F(G))$, if summation of weights of all edges that are incident to v in $F(G)$ equals to σ, then v is called a *leaf node* (that is a degree-1 vertex in a tree or a degree-2 vertex in a cycle). Note that, for any leaf node v, $\rho^{-1}(v)$ is a leaf of G, conversely, for any leaf L of G, $\rho(L)$ is a leaf node of $F(G)$. It is shown that $F(G)$ can be constructed in $O(|V||E|)$ time [11] or in $O(\sigma^2 |V| \log(|V|/\sigma) + |E|)$ time [7].

Two edges e_1, e_2 are said to be *independent* if and only if $V(e_1) \cap V(e_2) = \emptyset$, and a set $Q \subseteq E$ is called an *independent set* or a *matching* of G if and only if any pair of edges in Q are independent. An independent set of maximum cardinality in G is called a *maximum matching* of G.

Proposition 1. *[5] For distinct sets $X, Y \subset V$ of any graph $G = (V, E)$,*

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2|(V - X \cup Y, X \cap Y)|,$$

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2|(X - Y, Y - X)|.$$

Let $\lceil x \rceil$ ($\lfloor x \rfloor$, respectively) denote the minimum integer no smaller (the maximum one no greater) than x.

2.2 σ-Components and leaf-graphs

Let $\lambda(G) = \sigma > 0$. Let X_1, X_2 be distinct $(\sigma+1)$-components of G. The pair $\{X_1, X_2\}$ are called an *adjacent pair* (denoted as $X_1 \chi X_2$) if any two vertices $w \in X_1$ and $w' \in X_2$ are adjacent in G, or called a *nonadjacent pair* (denoted as $X_1 \overline{\chi} X_2$) otherwise. Let

$$V_C = \{v | v \text{ represents an individual } (\sigma+1)\text{-component of } G\}$$

and let $S(v) \in \Gamma_G(\sigma+1)$ denote the one represented by $v \in V_C$. Let $C(G) = (V_C, E_C)$ be defined by V_C and $E_C = \{(v, v') | v, v' \in V_C \text{ and } S(v) \overline{\chi} S(v')\}$, and it is called the *component graph* of G. Let $LF(G) = \{X \in \Gamma_G(\sigma+1) | X \text{ is a leaf of } G\}$ and $V_R = \{v | v \text{ represents an individual leaf of } G\} \subseteq V_C$. Let $Y(v)$ denote the leaf $(\sigma+1)$-component represented by $v \in V_R$. Let $R(G) = (V_R, E_R)$ be the subgraph of $C(G)$ defined by $E_R = \{(v, v') \in E_C | v, v' \in V_R \text{ and } Y(v) \overline{\chi} Y(v')\}$, and it is called the *leaf-graph* of G.

Property 1. $R(G)$ is simple.

Let Y_i, $i = 1, 2, 3, 4$, be distinct leaves of G. A set of two nonadjacent pairs $\{Y_1, Y_2\}, \{Y_3, Y_4\}$ is called a *D-combination* if they are disjoint (that is, $\{Y_1, Y_2\} \cap \{Y_3, Y_4\} = \emptyset$). In general, for $2t$ distinct leaves Y_i, $i = 1, \ldots, 2t$, of G with $t \geq 2$, a set of t nonadjacent pairs $\{Y_1, Y_2\}, \ldots, \{Y_{2t-1}, Y_{2t}\}$ is called a *D-set* of G if any two pairs of the set form a D-combination. Let $Y_1 \chi \{Y_2, Y_3\}$ denote that both $Y_1 \chi Y_2$ and $Y_1 \chi Y_3$ hold. A D-combination $\{\{Y_1, Y_2\}, \{Y_3, Y_4\}\}$ is called an *I-combination* (denoted as $\{Y_1, Y_2\} \angle \{Y_3, Y_4\}$) if either $Y_1 \chi \{Y_3, Y_4\}$ or $Y_2 \chi \{Y_3, Y_4\}$ holds. If neither $\{Y_1, Y_2\} \angle \{Y_3, Y_4\}$ nor $\{Y_3, Y_4\} \angle \{Y_1, Y_2\}$ holds then we denote $\{Y_1, Y_2\} \not\angle \{Y_3, Y_4\}$.

We first show some basic results on $R(G)$ and leaves of G.

Proposition 2. *Suppose that G is simple. Then either $|Y| = 1$ or $|Y| \geq \sigma+2$ for any $Y \in LF(G)$.*

Since each leaf Y has $d_G(Y) = \sigma$, we obtain the next proposition by Proposition 2.

Proposition 3. *Suppose that G is simple. If $\{Y_1, Y_2\} \subseteq LF(G)$ is an adjacent pair then $|Y_1| = |Y_2| = 1$.*

Proposition 4. $d_{R(G)}(v) \geq \max\{|V_R| - (\sigma + 1), 0\}$ *for any $v \in V_R$.*

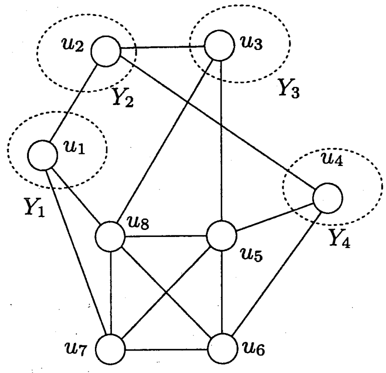

Fig. 1. A simple graph G with $\lambda(G) = 3$ and $|LF(G)| = 4$.

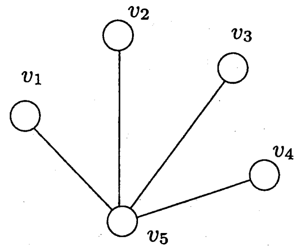

Fig. 2. A structural graph $F(G)$ of G in Fig. 1, where all edge-weights are 3 and none of them are written. In this case leaves Y_i in $LF(G)$ of the graph G shown in Fig. 1 are represented as nodes v_i of $F(G)$ for $i = 1, \ldots, 5$: it may happen that G has a node to which no corresponding leaf of $LF(G)$ exists.

2.3 Examples

Let $G = (V, E)$ with $|V| \geq \sigma+2$ and $\lambda(G) = \sigma$ be any given simple graph. Let $OPT(M)$ or $OPT(S)$ denote the cardinality of an optimum solution to $(\sigma+1)$ECA(*,MA) or to $(\sigma+1)$ECA(S,SA) for G, respectively. For $\sigma = 3$, we give an example such that $OPT(S) = OPT(M) + 1$. For the graph G with $|LF(G)| = 4$ shown Fig. 1, $R(G)$ is given in Fig. 3. The set of edges $\{(u_1, u_3), (u_2, u_4)\}$ is an optimum solution to 4ECA(*,MA), while $\{(u_1, u_3), (u_2, u_8), (u_3, u_7)\}$ is an optimum solution to 4ECA(S,SA) and, therefore, $OPT(S) = 3 = OPT(M) + 1$.

3 Maximum matchings of leaf-graphs

One of requirements in finding a solution to $(\sigma+1)$ECA(S,SA) or $(\sigma+1)$ECA(*,SA) with $\sigma \geq 1$ is to obtain a largest D-set. Hence, in this section, the cardinality of a maximum D-set is investigated by considering a maximum matching $\mathcal{M}$ of $R(G)$.

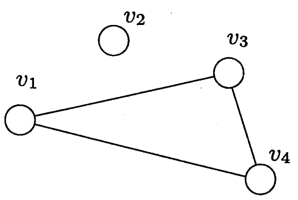

Fig. 3. The leaf-graph $R(G)$ of G in Fig. 1.

Let $\mathcal{M}$ denote any fixed maximum matching of $R(G)$ in the following discussion unless otherwise stated, where we assume that $\lambda(G) = \sigma \geq 1$.

Proposition 5. *$|\mathcal{M}|$ satisfies one of the following (1)–(3).*

(1) If $|V_R| \geq 2\sigma + 1$ or if σ is even and $|V_R| = 2\sigma$ then $|\mathcal{M}| = \lfloor |V_R|/2 \rfloor$.

(2) If σ is odd and $|V_R| = 2\sigma$ then

$$\lfloor |V_R|/2| \rfloor - 1 \leq |\mathcal{M}| \leq \lfloor |V_R|/2 \rfloor.$$

(3) If $|V_R| \leq 2\sigma - 1$ then

$$\max\{0, \min\{|V_R| - \sigma, \lfloor |V_R|/2 \rfloor\}\} \leq |\mathcal{M}| \leq \lfloor |V_R|/2 \rfloor.$$

Corollary 1. *Suppose that $|V_R| = 2\sigma$ and $\sigma = 2m + 1$. If $|\mathcal{M}| = \lfloor |V_R|/2 \rfloor - 1$ then $G = (V, E)$ is a complete bipartite graph with $V = X \cup Y$, $X \cap Y = \emptyset$, $|X| = |Y| = \sigma$ and $E = \{(x, y) | x \in X, y \in Y\}$.*

The relationship among G, $C(G)$ and $R(G)$ shows the following proposition concerning $|V_R|$, $|\mathcal{M}|$ and $|E'|$ of any optimum solution E' to $(\sigma + 1)$ECA(S,SA).

Proposition 6. *Let E' be any solution to G in $(\sigma+1)$ECA(S,SA) and $\mathcal{M}$ be a maximum matching of $R(G)$. Then*

$$|V_R| - |\mathcal{M}| \leq |E'|. \tag{3.1}$$

4 Augmentation by edge-interchange

We explain an operation called edge-interchange which was originally introduced in [21, 22] for an efficient augmentation. It is also used in [14–18]. Let $LF(G) = \{Y_1, \ldots, Y_q\}$ ($q = |LF(G)|$) denote the class of all leaves of G and choose $y_i \in Y_i$ as the representative of Y_i. Let

$$Y(G) = \{y_i | Y_i \in LF(G)\}, \quad q \geq 2, \text{ and } r = \lceil q/2 \rceil.$$

We can easily prove the next proposition.

Proposition 7. *If there is a set E' of edges, each connecting vertices of G, such that $E' \cap E = \emptyset$ and $V(E') = Y(G) \subseteq S$ for some $(\sigma + 1)$-component S of $G + E'$, then $S = V$.*

Let Y stand for $Y(G)$ in the rest of the section.

4.1 Attachments

We have $d_G(Y_i) = \sigma$ and $\lambda(y_i, y_j; G) = \sigma$ for any $y_i, y_j \in Y$ ($i \neq j$). An edge set F is called an *attachment* (for G) if and only if the following (1) through (4) hold:

(1) $V(F) \subseteq Y$,
(2) $F \cap E(G) = \emptyset$,
(3) $V(e) \neq V(e')$ ($\forall e, e' \in F, e \neq e'$), and
(4) if q ($= |LF(G)|$) is odd then F has at most one pair f, f' such that $|V(f) \cap V(f')| = 1$; or if q is even then F has no such pair.

Let F be any attachment for G. For each $e = (u, v) \in F$, $G + F$ has a new $(\sigma + 1)$-component, denoted by $\mathcal{A}(e, G + F)$, containing $V(e)$.

We are going to show that we can find a minimum attachment $Z(\sigma + 1) = \{e_1, \ldots, e_r\}$ ($r = \lceil q/2 \rceil$) such that $\lambda(G + Z(\sigma + 1)) = \sigma + 1$. Although there are two cases: $r = 1$ and $r \geq 2$, we discuss only the latter case in the following. (Note that if $r = 1$ then we immediately obtain the desired attachment F.)

4.2 Finding a minimum attachment

Suppose that there are an attachment F for G and vertices $y_{ij} \in Y - V(F)$, $1 \leq i, j \leq 2$, where y_{11}, y_{12}, y_{21} are distinct, and if y_{22} is equal to one of the other three then we assume that $y_{22} = y_{21}$ (see Fig. 4). We use the following notations:

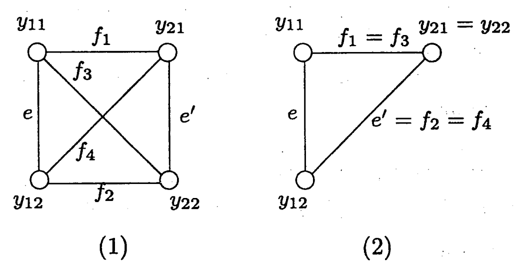

Fig. 4. The edges e, e' and f_i, $1 \leq i \leq 4$: (1) $y_{21} \neq y_{22}$; (2) $y_{21} = y_{22}$.

$$L = G + F, \quad e = (y_{11}, y_{12}),$$

$$e' = \begin{cases} (y_{21}, y_{22}) & \text{if } y_{21} \neq y_{22} \\ (y_{12}, y_{21}) & \text{if } y_{21} = y_{22}, \end{cases}$$

$$\mathcal{A}(e) = \mathcal{A}(e, L+\{e, e'\}), \quad \mathcal{A}(e') = \mathcal{A}(e', L+\{e, e'\}),$$

$$f_1 = (y_{11}, y_{21}), \quad f_2 = (y_{12}, y_{22}),$$

$$f_3 = (y_{11}, y_{22}), \quad f_4 = (y_{12}, y_{21}),$$

where we set $f_1 = f_3$ and $e' = f_2 = f_4$ if $y_{21} = y_{22}$, and

$$\mathcal{A}(f_i) = \begin{cases} \mathcal{A}(f_i, L + \{f_1, f_2\}) & \text{if } 1 \leq i \leq 2 \\ \mathcal{A}(f_i, L + \{f_3, f_4\}) & \text{if } 3 \leq i \leq 4. \end{cases}$$

Note that $e, e', f_i \notin E(L), 1 \leq i \leq 4$. We have the following two cases.

Case I: $\mathcal{A}(e) \cap \mathcal{A}(e') = \emptyset$; Case II: $\mathcal{A}(e) \cap \mathcal{A}(e') \neq \emptyset$ (that is, $\mathcal{A}(e) = \mathcal{A}(e')$).

For Case I, we are going to show that there are two edges f, f', with $V(f) \cup V(f') = V(e) \cup V(e')$, such that

$$\mathcal{A}(e) \cup \mathcal{A}(e') \subseteq \mathcal{A}(f, L+\{f, f'\}) = \mathcal{A}(f', L+\{f, f'\}).$$

That is, we can add two edges so that one $(\sigma+1)$-component containing $\mathcal{A}(e) \cup \mathcal{A}(e')$ may be obtained. Finding and adding such a pair of edges f, f' is called *edge-interchange* (with respect to $V(e_1) \cup V(e_2)$).

Suppose that $\mathcal{A}(e) \cap \mathcal{A}(e') = \emptyset$. Note that $y_{21} \neq y_{22}$ in this case. Let K be any fixed $(\mathcal{A}(e), \mathcal{A}(e'))$-cut of $L + \{e, e'\}$, and let B_i, $1 \leq i \leq 2$, denote the two sets of vertices in $L + \{e, e'\}$ such that $B_1 \cup B_2 = V$, $B_2 = V - B_1$, $K = (B_1, B_2; L + \{e, e'\})$, $\mathcal{A}(e) \subseteq B_1$ and $\mathcal{A}(e') \subseteq B_2$. $|K| = \sigma = \lambda(y_1, y_2; L'')$ for any $y_i \in B_i$, $1 \leq i \leq 2$, where L'' denotes L, $L + e$, $L + e'$ or $L + \{e, e'\}$. K is a (y_1, y_2)-cut of L. Suppose that f and f' satisfy either (i) or (ii):

(i) $f = f_1$, $f' = f_2$, or (ii) $f = f_3$, $f' = f_4$, where $\{f, f'\} \cap E(L) = \emptyset$.

The next proposition shows a property of edge-interchange.

Proposition 8. *If* $\mathcal{A}(e) \cap \mathcal{A}(e') = \mathcal{A}(f_1) \cap \mathcal{A}(f_2) = \emptyset$ *then* $\mathcal{A}(f_3) \cap \mathcal{A}(f_4) \neq \emptyset$, *that is,* $\mathcal{A}(f_3) = \mathcal{A}(f_4)$.

Let $\{f, f'\}$ denote the following pair of edges:

$\{e, e'\}$ if $\mathcal{A}(e) = \mathcal{A}(e')$(the case with $V(e) \cap V(e') = \emptyset$ is included);

$\{f_1, f_2\}$ if $\mathcal{A}(e) \cap \mathcal{A}(e') = \emptyset$ and $\mathcal{A}(f_1) = \mathcal{A}(f_2)$;

$\{f_3, f_4\}$ if $\mathcal{A}(e) \cap \mathcal{A}(e') = \mathcal{A}(f_1) \cap \mathcal{A}(f_2) = \emptyset$.

Clearly, $\{f, f'\} \cap E(L) = \emptyset$. Such a pair f, f' are called an *augmenting pair* (with respect to $\{y_{11}, y_{12}, y_{21}, y_{22}\}$) of L.

Corollary 2. *Let* $L' = L + \{f, f'\}$ *for any augmenting pair* f, f'. *Then* $L' - f'$ *has no* σ*-cut separating* $V(f')$ *from* $V(f)$. *That is, if* $L' - f'$ *has a* σ*-cut* K *separating a vertex of* $V(f')$ *from* $V(f)$ *then* K *separates the two vertices of* $V(f')$.

From Corollary 2, other important properties (Proposition 9–11) of edge-interchange are obtained.

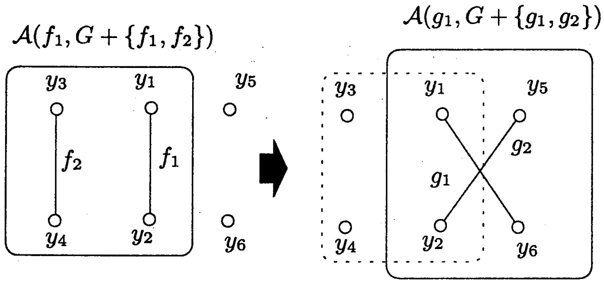

Fig. 5. The two $(\sigma + 1)$-components $\mathcal{A}(f_1, G + \{f_1, f_2\})$ and $\mathcal{A}(g_1, G + \{g_1, g_2\})$ produced by two augmenting pairs $\{f_1, f_2\}$ and $\{g_1, g_2\}$, respectively.

Proposition 9. *Suppose that* G *has six leaves* $Y_i \in LF(G)$ $(1 \leq i \leq 6)$, *and choose* $y_i \in Y_i$ *as a representative of each* Y_i. *Suppose that* $\{f_1, f_2\}$ *is an augmenting pair with respect to* $\{y_i | 1 \leq i \leq 4\}$ *of* G. *If* $\mathcal{A}(f_1, G+\{f_1, f_2\})$ *is a leaf then, for each* $i \in \{1, 2\}$, *there is an augmenting pair* $\{g_1, g_2\}$ *with respect to* $V(f_i) \cup \{y_5, y_6\}$ *of* G *such that* $\mathcal{A}(g_1, G + \{g_1, g_2\})$ *is not a leaf (see Fig. 5).*

By Proposition 9, we obtain the following procedure that is a modified version of the procedure given in [15]. It finds a sequence of edges $e_1, \ldots, e_r$ $(r = \lceil |LF(G)|/2 \rceil \geq 1)$ by repeating edge-interchange operation, where handling the case with $|LF(G)| = 2$ is included. Note that edges with which we are concerned are those connecting vertices belonging to distinct leaves. If an edge g connects a vertex in a leaf Y_i and another vertex in a leaf Y_j $(i \neq j)$ then, for simplicity, we say that g *connects* Y_i *and* Y_j.

Procedure *FIND_EDGES;*
begin
1. $G_1 \leftarrow G$; $\pi \leftarrow LF(G)$; $i \leftarrow 1$; $E'_1 \leftarrow \emptyset$;
2. **while** $\pi \neq \emptyset$ **do**
begin
3. **if** $|\pi| = 2$ **then**
4. $f_i \leftarrow$an edge connecting the two leaves of π; $E''_i \leftarrow \{f_i\}$;
5. **else if** $|\pi| \leq 5$ **then**
6. Find an augmenting pair $E''_i = \{f_i, f'_i\}$ by Proposition 8;
7. **else** /* $|\pi| \geq 6$ */
8. Find an augmenting pair $E''_i = \{f_i, f'_i\}$ by Proposition 9;
9. $E'_{i+1} \leftarrow E'_i \cup E''_i$; $G_{i+1} \leftarrow G_i + E''_i$; $\pi \leftarrow \pi - \{Y(v) | v \in V(E''_i)\}$; $i \leftarrow i + 1$;
end
end;

Proposition 10. *G_{i+1} has a leaf containing $\mathcal{A}(f_i, G_{i+1})$ if and only if $|LF(G_i)| = 5$ just after the execution of Step 9 in FIND_EDGES.*

Note that executing Step 6 or Step 8 once can be done in $O(|V_R|)$ by using a structural graph $F(G)$, and we can construct $F(G)$ in $O(\sigma^2|V|\log(|V|/\sigma) + |E|)$ time (see [7]). The details are omitted here.

The next proposition holds for the edge set E' produced by *FIND_EDGES*.

Proposition 11. *Let $Z(\sigma+1) = \{e_1, \ldots, e_r\}$ ($r = \lfloor |LF(G)/2 \rfloor$) be given by FIND_EDGES. Then $Z(\sigma+1)$ is a minimum attachment such that $\lambda(G') = \sigma+1$, where $G' = G + Z(\sigma+1)$. Furthermore the procedure runs in $O(\sigma^2|V|\log(|V|/\sigma) + |E| + |V_R|^2)$ time.*

5 $(\sigma+1)$ECA(S,SA) for G having less than $2\sigma+6$ leaves

We denote $LF(G) = \{Y_i | 1 \leq i \leq q\}$ ($q = |LF(G)|$), $Y(G) = \{y_1, \ldots, y_q\}$ and $V_R = \{v_1, \ldots, v_q\}$, where each y_i is represented as v_i in $R(G)$. First we consider the case where G has two or three leaves.

Proposition 12. *If $q = 2$ then the following (1) or (2) holds.*

(1) If $Y_1\overline{\chi}Y_2$ then $|\mathcal{M}| = 1$, there are two vertices $y_i \in Y_i$, $i = 1, 2$, such that $E' = \{(y_1, y_2)\}$ is a solution, and $OPT(S) = OPT(M) = 1$.

(2) If $Y_1\chi Y_2$ then $|\mathcal{M}| = 0$, there are three vertices $y_i \in Y_i$ ($i = 1, 2$), $x \in V - (Y_1 \cup Y_2)$ such that $E' = \{(y_1, x), (y_2, x)\}$ is a solution, and $OPT(S) = 2 = OPT(M) + 1$.

Proposition 13. *If $q = 3$ and there exist two leaves Y_1, Y_2 with $Y_1\overline{\chi}Y_2$ then $|\mathcal{M}| = 1$, there are distinct edges e_1, e_2 such that $E' = \{e_1, e_2\}$ is a solution, and $OPT(S) = OPT(M) = 2$.*

Next we consider the remaining case where $3 \leq q < 2\sigma + 6$. For each $e' = (x', y') \in \mathcal{M}$, we can choose two vertices $x \in Y(x')$, $y \in Y(y')$, and let $e = (x, y)$ be an edge, which is not included in E. We fix such an edge e for each $e' \in \mathcal{M}$, and let

$$E'_0 = \{e = (x, y) \mid (x', y') \in \mathcal{M}\}.$$

Proposition 14. $|E'_0| = |\mathcal{M}|$ *and* $E'_0 \cap E = \emptyset$.

In the rest of this section, we consider the graph $G + E'_0$. First we define two sets $\mathcal{L}_1$ and $\mathcal{K}_1$ as follows.

Let $G_1 = G + E'_0$ and let $\mathcal{L}_1$ be the set of new leaves of G_1 created by adding E'_0 to G. Clearly $|\mathcal{L}_1| \leq |\mathcal{M}|$. Let $\mathcal{K}_1 = LF(G + E'_0) - \mathcal{L}_1$ ($\subseteq LF(G)$). Since $\mathcal{M}$ is a maximum matching of $R(G)$, Proposition 3 shows that each leaf in $\mathcal{K}_1$ consists of only one vertex and that the set of vertices $\mathcal{K}'_1 = \{x \mid \{x\} \in \mathcal{K}_1\}$ induces a complete graph of G and of $G + E'_0$.

We are going to propose an $O(\sigma^2|V|\log(|V|/\sigma) + |E| + |V_R|^2)$ time algorithm such that it finds an optimum solution if $|\mathcal{L}_1| \leq |\mathcal{K}_1|$ and such that a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1| > |\mathcal{K}_1|$. Note that we have $|\mathcal{L}_1| \leq |\mathcal{K}_1|$ if $|\mathcal{M}| \leq \lfloor |V_R|/3 \rfloor$.

Proposition 15. *Let $\{y'_1\}, \{y'_2\} \in \mathcal{K}_1$ ($y'_1 \neq y'_2$) and $Y_1, Y_2 \in \mathcal{L}_1$ ($Y_1 \neq Y_2$). If $\{(y_1, y'_1), (y_2, y'_2)\}$ is not an augmenting pair with $y_1 \in Y_1$ and $y_2 \in Y_2$ then there are $y_3 \in Y_1$ and $y_4 \in Y_2$ such that $\{(y_4, y'_1), (y_3, y'_2)\}$ is an augmenting pair and $(y_4, y'_1), (y_3, y'_2) \notin E$ (See Fig. 6).*

We obtain the next proposition by Propositions 9 and 15.

Proposition 16. *Assume that $|\mathcal{L}_1| \geq 3$ and $|\mathcal{K}_1| \geq 3$. Then there exists an augmenting pair $\{f_1, f_2\}$ such that $f_1 = (y_1, y'_1) \notin E \cup E'_0$, $f_2 = (y_2, y'_2) \notin E \cup E'_0$, $\{\{y'_1\}, \{y'_2\}\} \subseteq \mathcal{K}_1$ ($y'_1 \neq y'_2$), $\mathcal{L}_1$ has two distinct sets Y_1, Y_2 with $y_1 \in Y_1$, $y_2 \in Y_2$*

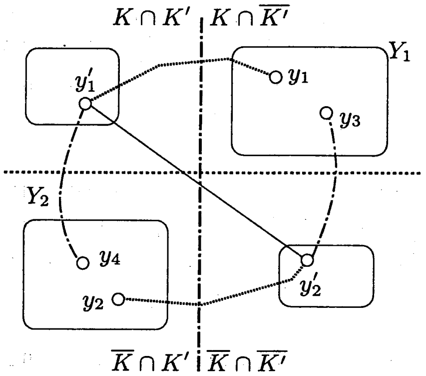

Fig. 6. A situation for Proposition 15

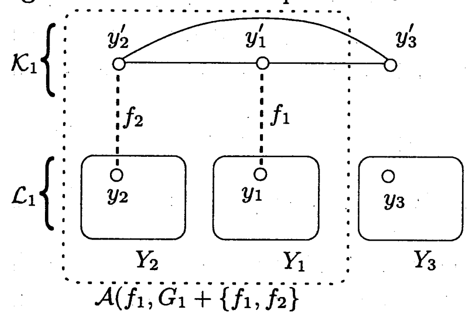

Fig. 7. $\mathcal{A}(f_1, G + \{f_1, f_2\})$ in the proof of Proposition 16

and $\mathcal{A}(f_1, G+\{f_1, f_2\})$ is not a leaf. Furthermore $\mathcal{L}_1 \cup \mathcal{K}_1 - \{\{y_1'\}, \{y_2'\}\}, Y_1, Y_2\}$ is the set of all leaves in $G_1 + \{f_1, f_2\}$. (See Fig. 7)

Next we are going to discuss the case where $|\mathcal{L}_1| \leq 2$ or $|\mathcal{K}_1| \leq 2$.

Proposition 17. *Suppose that $|\mathcal{L}_1| \leq 2$ and $|\mathcal{L}_1| \leq |\mathcal{K}_1|$. Then there exists a set $E_2' = \{f_1, \ldots, f_{|\mathcal{K}_1|}\}$ such that $\lambda(G_1 + E_2') \geq \sigma + 1$ and $E_2' \cap (E \cup E_0') = \emptyset$.*

It remains to consider the cases ($|\mathcal{L}_1| \geq 3$ and $|\mathcal{K}_1| \leq 2$) and ($|\mathcal{L}_1| \leq 2$ and $|\mathcal{L}_1| > |\mathcal{K}_1|$), for which the next proposition holds.

Proposition 18. *Suppose that one of the following (1)-(3) holds: (1) $|\mathcal{L}_1| \geq 3$ and $|\mathcal{K}_1| \leq 2$; (2) $|\mathcal{L}_1| = 2$ and $|\mathcal{K}_1| = 1$; (3) $|\mathcal{L}_1| = 2$ and $|\mathcal{K}_1| = 0$. Let $q_1 = |LF(G_1)|$ and $r_1 = \lceil \frac{q_1}{2} \rceil$. Then there exists a set $E_2'' = \{f_1, \ldots, f_{r_1}\}$ such that $\lambda(G_1 + E_2'') \geq \sigma + 1$ and $E_2'' \cap (E \cup E_0') = \emptyset$.*

The discussion from Propositions 16 through 18 is summarized in the following procedure *FIND_EDGES2*.

Procedure *FIND_EDGES2;*

begin
1. $G_0 \leftarrow G$; $\pi \leftarrow LF(G)$; $E_0' \leftarrow \emptyset$; $\rho \leftarrow \emptyset$;
2. Find an edge set E_0' as in Proposition 14;
   $G_1 \leftarrow G_0 + E_0'$;
   Determine $\mathcal{L}_1$ and $\mathcal{K}_1$; $i \leftarrow 1$;
3. **while** $\mathcal{K}_i \neq \emptyset$ **do**
   begin
4. **if** $|\mathcal{L}_i| \geq 3$ and $|\mathcal{K}_i| \geq 3$ **then**
   Find an augmenting pair $\{f, f'\}$
   by Proposition 16; $E_i'' \leftarrow \{f, f'\}$;
5. **else if** $|\mathcal{L}_i| \leq 2$ and $|\mathcal{L}_i| \leq |\mathcal{K}_i|$ **then**
   Find an edge set E_i'' by Proposition 17;
6. **else**
   Find an edge set E_i'' by Proposition 18;
7. Construct $\mathcal{K}_{i+1}$ and $\mathcal{L}_{i+1}$; $E_i' \leftarrow E_{i-1}' \cup E_i''$;
   $G_{i+1} \leftarrow G_i + E_i''$; $i \leftarrow i + 1$;
   end;
8. **if** $\lambda(G_i) = \sigma$ **then**/* the case with $|\mathcal{L}_i| \neq 0$ */
   Find an edge set E_i'' by Proposition 18;
   $E_{i+1}' \leftarrow E_{i-1}' \cup E_i''$;

end;

Proposition 19. *FIND_EDGES2 produces an optimum solution if $|\mathcal{L}_1| \leq |\mathcal{K}_1|$.*

Proposition 20. *FIND_EDGES2 gives a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1| > |\mathcal{K}_1|$.*

Remark 1. Let $\mathcal{M}$ be any maximum matching of $R(G)$. If $|\mathcal{M}| \leq \lfloor \frac{|LF(G)|}{3} \rfloor$ then $|\mathcal{L}_1| \leq |\mathcal{K}_1|$ and we can find an optimum solution in polynomial time. If $\lfloor \frac{|LF(G)|}{3} \rfloor < |\mathcal{M}| \leq \lfloor \frac{|LF(G)|}{2} \rfloor$ then $|\mathcal{L}_1| \leq |\mathcal{K}_1|$ or $|\mathcal{L}_1| > |\mathcal{K}_1|$. Since the proof of NP-completeness of kECA(S,SA) in [10] is given for the case with $|\mathcal{M}| = \lfloor \frac{|LF(G)|}{2} \rfloor$, we consider approximate solutions if $|\mathcal{L}_1| > |\mathcal{K}_1|$.

Theorem 1. *Suppose that $|LF(G)| \leq 2\sigma + 6$. Then FIND_EDGES2 can find an optimum solution if $|\mathcal{L}_1| \leq |\mathcal{K}_1|$, or a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1| > |\mathcal{K}_1|$, in $O(\sigma^2|V|\log(|V|/\sigma) + |E|)$ time.*

6 $(\sigma + 1)$ECA(S,SA) for G having at least $2\sigma + 6$ leaves

In this case, Proposition 5(3) shows that any maximum matching $\mathcal{M}$ of $R(G)$ has $|\mathcal{M}| = \lfloor \frac{|LF(G)|}{2} \rfloor$. First, some basic results on nonadjacent pairs and edge interchange operation are going to be given.

Proposition 21. *Suppose that there are a non-adjacent pair of leaves $Y_1, Y_2 \in LF(G)$ and two vertices $y_i \in Y_i$, $i = 1, 2$, with $(y_1, y_2) \notin E$, such that $G' = G + \{(y_1, y_2)\}$ has a leaf S containing $Y_1 \cup Y_2$. Let $\mathcal{L}' = \{Y \subseteq S | Y \in \Gamma_G(\sigma + 1)\}$, $X = Y_1 \cup Y_2$ and $Z = \bigcup_{Y \in LF(G)-\{Y_1,Y_2\}} Y$. Then $|(X, Z; G)| \leq \sigma - 1$ if $|\mathcal{L}'| \geq 3$.*

The next proposition can be proved by using Propositon 21.

Proposition 22. *Suppose $\sigma \geq 3$ and let $\mathcal{M}' = \{(v_{2i-1}, v_{2i}) | 1 \leq i \leq m\} \subseteq \mathcal{M}$ for some $m \leq |\mathcal{M}|$, and put $Y_j = Y(v_j)$ for each $v_j \in V_R$.*

(1) If $|\mathcal{M}'| \geq 2$ and there are distinct indices i, j with $1 \leq i, j \leq m$ such that $\{Y_{2i-1}, Y_{2i}\} \nparallel \{Y_{2j-1}, Y_{2j}\}$ then (i) and (ii) hold.

(i) These leaves are partitioned into a D-combination $\{\{L'_1, L'_2\}, \{L'_3, L'_4\}\}$ having four vertices $y_t \in L'_t$, $t = 1, 2, 3, 4$, such that $G + \{(y_1, y_2), (y_3, y_4)\}$ has a $(\sigma + 1)$-component S containing all L'_t, $t = 1, 2, 3, 4$.

(ii) The $(\sigma + 1)$-component S' of $G + \{(y_1, y_2)\}$ such that $L'_1 \cup L'_2 \subseteq S'$ is not a leaf.

(2) If $|\mathcal{M}'| \geq \lceil \sigma/2 \rceil + 1$ and no such pair of indices as in (1) exist then, for each $(v_{2i-1}, v_{2i}) \in \mathcal{M}'$, there are vertices $y_{2i-1} \in Y_{2i-1}$ and $y_{2i} \in Y_{2i}$ such that $G' = G + \{(y_{2i-1}, y_{2i})\}$ is a simple graph having a $(\sigma + 1)$-component X which is not a leaf and which contains $Y_{2i-1} \cup Y_{2i}$.

Proposition 23. *Suppose that there is a set $\mathcal{M}' = \{(v_{2i-1}, v_{2i}) | 1 \leq i \leq m\} \subseteq \mathcal{M}$ for some m with $\sigma + 2 \leq m \leq |\mathcal{M}|$, and put $Y_i = Y(v_i)$ for each $v_i \in V_R$. Then there is an edge $(v_{2h-1}, v_{2h}) \in \mathcal{M}'$ with $\{Y_1, Y_2\} \nparallel \{Y_{2h-1}, Y_{2h}\}$.*

By combining Propositions 9, 22 and 23, we obtain the following proposition.

Proposition 24. *Suppose that there is a set $\mathcal{M}' = \{f_i = (v_{2i-1}, v_{2i}) | 1 \leq i \leq m\} \subseteq \mathcal{M}$ for some m with $\sigma + 3 \leq m \leq |\mathcal{M}|$, and put $Y_i = Y(v_i)$ for each $v_i \in V_R$. Then there exists an augmenting pair $\{e'_1, e'_2\}$ with respect to $Y_1, Y_2, Y_{2j-1}, Y_{2j}$ such that $G + \{e'_1, e'_2\}$ is simple and has no leaf S with $Y_1 \cup Y_2 \cup Y_{2j-1} \cup Y_{2j} \subseteq S$, where $\{f_1, f_j\} \subseteq \mathcal{M}'$.*

Based on Proposition 24, the next procedure *FIND_EDGES3* is obtained.

Procedure *FIND_EDGES3;*
  begin
1. $G_1 \leftarrow G$; $\pi \leftarrow LF(G)$; $i \leftarrow 1$; $E'_0 \leftarrow \emptyset$;
2. **while** $\pi \neq \emptyset$ **do**
   begin
3. **if** $|\pi| \leq 3$ **then**
4. Find an edge set E''_i as E' in Proposition 12(1) or 13;
5. **else**
   begin /* $|\pi| \geq 4$ */
6. Find a matching $\mathcal{M}'' = \{(v_{2p-1}, v_{2p}) | 1 \leq p \leq m'\}$ of $R(G_i)$, where if $|\pi| \leq 2\sigma + 6$ then $m' \leftarrow \lfloor \pi/2 \rfloor$, otherwise $m' \leftarrow \sigma + 3$;
7. **if** $|\pi| \leq 2\sigma + 6$ **then**
   begin
   Choose $E'_s \subseteq E'_i$ with $|E'_s| = \sigma + 3 - m'$ appropriately;
   $\mathcal{M}' \leftarrow \mathcal{M}'' \cup \{(v, w) \in E_R | (v', w') \in E'_s, v' \in Y(v), w' \in Y(w)\}$;
   /* $\mathcal{M}'$ is a matching on $R(G)$ in the case.*/
   end;
   else
   $\mathcal{M}' \leftarrow \mathcal{M}''$;
8. Find an augmenting pair E''_i as $\{e'_1, e'_2\}$ in Proposition 24 by choosing $f_1 \in \mathcal{M}''$;
   /* Note that $|\mathcal{M}'| = \sigma + 3$. */
9. **if** $f_j \in \mathcal{M}' - \mathcal{M}''$ for f_j of Proposition 24 **then**
   begin /* In the case with $|\pi| \leq 2\sigma + 6$ */
   $E'_i \leftarrow E'_i - \{(y_{2j-1}, y_{2j})\}$,
   $G_i \leftarrow G_i - \{(y_{2j-1}, y_{2j})\}$, where $y_{2j-1} \in Y_{2j-1}$ and $y_{2j} \in Y_{2j}$;
   end;
10. $E'_{i+1} \leftarrow E'_i \cup E''_i$; $G_{i+1} \leftarrow G_i + E''_i$;
   $\pi \leftarrow \pi - \{Y(v) | v \in V(E''_i)\}$; $i \leftarrow i + 1$;
   end;
  end;

Proposition 25. *Any set E'_i finally obtained at the termination of FIND_EDGES3 is a minimum attachment such that $\lambda(G') = \sigma + 1$, where $G' = G + E'$.*

Theorem 2. *If G has at least $2\sigma + 6$ leaves then the algorithm FIND_EDGES3 correctly finds a solution E' to $(\sigma + 1)$ECA(S,SA) for any given G with $\lambda(G) = \sigma$ in $O(\sigma^2 |V| \log(|V|/\sigma) + |E| + |V_R|^2)$ time.*

7 Concluding Remarks

The paper has proposed

(1) an $O(\sigma^2|V|\log(|V|/\sigma)+|E|+|V_R|^2)$ time algorithm for finding an optimum solution if G has at least $2\sigma+6$ leaves or if $|\mathcal{L}_1|\leq|\mathcal{K}_1|$ and G has less than $2\sigma+6$ leaves,

(2) an $O(\sigma^2|V|\log(|V|/\sigma)+|E|)$ time one for a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1|>|\mathcal{K}_1|$ and G has less than $2\sigma+6$ leaves.

We can improve the first algorithm to an $O(\sigma^2|V|\log(|V|/\sigma)+|E|)$ time one by devising how to check whether or not $\{f_1,f_2\}$ is an augmenting pair, and whether or not $\mathcal{A}(f_1,G+\{f_1,f_2\})$ is a leaf in Proposition 9.

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